1. (a) Find the value of the parameter $m$ for which the sum of the squares of the roots of the equation $x^2 + (m-2)x - (m+3) = 0$ has the least value.

   (b) Let $\alpha, \beta$ be the real roots of the equation $ax^2 + bx + c = 0$ and let $\alpha^n + \beta^n = S_n$ for $n \geq 1$, then find the value of the determinant of $A$, where

$$A = \begin{bmatrix} 3 & 1+S_1 & 1+S_2 \\ 1+S_1 & 1+S_2 & 1+S_3 \\ 1+S_2 & 1+S_3 & 1+S_4 \end{bmatrix}.$$

[5 + 10 = 15]

2. (a) Prove that if $x > 0, y > 0$ and $x + y = 1$, then $(1 + \frac{1}{x})(1 + \frac{1}{y}) \geq 9$.

   (b) Given that the definite integral $\int_0^3 \frac{dx}{\sqrt[3]{(x-2)}}$ has a finite value, find the value of the integral. [5 + 10 = 15]

3. (a) Suppose $\alpha, \beta$ are such that $0 < \alpha < \beta < \frac{\pi}{2}$. Prove that there exists $\theta$ such that $\alpha < \theta < \beta$, for which

$$\frac{\sin\alpha - \sin\beta}{\cos\beta - \cos\alpha} = \cot\theta.$$

   (b) Ram and Shyam stand in a queue at random along with 10 other persons. Find the probability that there are exactly three persons between them. [8 + 7 = 15]

4. (a) Show that $\sqrt{C_1} + \sqrt{C_2} + \sqrt{C_3} + ... + \sqrt{C_n} \leq 2^{n-1} + \frac{n-1}{2}$ where $C_k = \begin{pmatrix} n \\ k \end{pmatrix}$.

   (b) Let $a > 0$ and $n \in \mathbb{N}$. Show that

$$\frac{a^n}{1 + a + a^2 + \ldots + a^{2n}} < \frac{1}{2n}.$$

[5 + 10 = 15]

5. (a) Prove that the rectangle of greatest area inscribed in a circle of radius 1 is a square. What is the area of the square?

   (b) A fair coin is tossed once. If it turns up heads, a balanced die is rolled twice. Then the sum of the number of dots on the faces that show up is recorded. If the coin turns up tails, the same die is rolled once. In this case the number of dots on the face that shows up is recorded. What is the probability that the recorded number is larger than 5? [7 + 8 = 15]

6. (a) Show that the sequence $\{S_n\}$ where $S_n = \frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{n+n}$ is convergent.

(b) Evaluate the value of $\int_1^2 x^{x^2+1} \, (2 \log x + 1) dx$.

[8 + 7 = 15]

7. (a) Evaluate the indefinite integral

$$\int \frac{\cos^3 x \ + \ \cos^5 x}{\sin^2 x \ + \ \sin^4 x} \, dx.$$

(b) For a fixed positive integer $n$, let

$$A = \begin{bmatrix} n! & (n+1)! & (n+2)! \\ (n+1)! & (n+2)! & (n+3)! \\ (n+2)! & (n+3)! & (n+4)! \end{bmatrix}$$

and $D = \text{Determinant}(A)$. Then show that

$$[\frac{D}{(n!)^3} - 4] \text{ is divisible by } n.$$

[7 + 8 = 15]

8. (a) Find the limit of the sequence $\{a_n\}$ given by

$$a_1 = 0, a_2 = \frac{1}{2} \text{ and } a_{n+1} = \frac{1}{3}(1 + a_n + a_{n-1}^3), \text{ for } n > 1.$$

(b) Let $\lambda$ and $\alpha$ be real numbers. Find the set of all values of $\lambda$ for which the system of linear equations

$$\begin{gathered} \lambda x + (\sin \alpha) y + (\cos \alpha) z = 0 \\ x + (\cos \alpha) y + (\sin \alpha) z = 0 \\ -x + (\sin \alpha) y - (\cos \alpha) z = 0 \end{gathered}$$

has a non-zero solution.

[9 + 6 = 15]